



## Overview of course

Course sessions:

- 1 Real and complex numbers and their functions (and a brisk run through the course contents)
- 2 Integral and differential calculus, in one dimension.
- 3 Matrices and vectors, including vector calculus, tensor analysis and the representation of rotations
- 4 Integral transforms: Fourier, Laplace, Mellin, Hankel
- 5 All you ever wanted to know about flashy sums but were scared to ask, including hints on using *Mathematica*
- 6 Probability and stochastic processes
- 7 Detection and estimation, including the principles of Kalman filtering
- 8 Clutter modelling – the K distribution and beyond
- 9 Simulation methods
- 10 Contour integration and Lagrange multipliers

Starting at 'rusty' A level, we will develop the tools needed to carry out radar performance calculations etc. and make sense of the related literature.

The exercises and discussion form an important part of the course. It must be stressed that these are not an excuse for the ritual humiliation of the ignorant and indolent or for bravura displays of mathematical technique. In addition to providing practice in the manipulation and interpretation of formulae the exercises are designed to simulate the process of carrying through 'real life' calculations. Consequently some will be straightforward and some will be really hard, just like life. In working through an exercise your principal aim should be to produce a reasonable solution in which you have confidence and can justify to your peers. How you do this is up to you. Do it from first principles, look it up in a book, confer with your colleagues, simulate it on the computer, plug it into *Mathematica*, do whatever you like. Giving up gracefully, as long as you can show how and why your proposed method of solution has failed, is a valid response (especially to an intractable problem) that will enable you to make an important contribution to the subsequent discussion. Remember that you are no longer at school: there are no marks or class lists. Ultimately the problems you address in your research will be both your teacher and your examiner; you are in competition only with yourself. One of the aims of this course is to prepare you for this rather daunting state of affairs.

## Motivation

Mathematics provides a remarkably compact, precise and powerful tool for the analysis of the physical world. Furthermore a little bit of mathematical know-how can be made to go a very long way. So, by becoming acquainted with a few new tricks, chosen for their relevance to radar research, you can access a large literature that would otherwise be rather impenetrable. Having done that you, at the very least, have a good idea of what all that squiggly stuff means. You will find that the techniques we cover are widely applicable and will stand you in good stead in your research; they provide a useful complement to simulation and data analysis studies. Maths is a common language for scientists and engineers, some proficiency in which enables you to cross cultural, interdisciplinary and linguistic barriers with ease. Even the Chinese do their sums in English.

The 'applied maths' of the sort we will cover here can be great fun; identifying, correctly posing and solving a practical problem is most gratifying. So the material covered in this course will help you enjoy your work and provide an extra element of job satisfaction.



In many situations a little maths adds a veneer of intellectual credibility to an otherwise quite ordinary product. With this course under your belt you have instant added value at your fingertips. The same applies to you, of course, as you present your way through life. A black belt in sums can come in very useful.

These and other reasons for studying maths are echoed in the opinions of the good and the great, of which a few are:

*'The analytical method I followed was no longer a bookish dogma, it was put to the test every day, it could be refined, made to conform to our aims, by a subtle play of reason, of trial and error. To make a mistake was no longer a vaguely comic accident that spoils an exam for you or affects your marks: to make a mistake was similar to when you went climbing - a contest, an act of attention, a step up that makes you more worthy and fit.'*

Primo Levi (1975)

*'We construct and keep on constructing, yet intuition is still a good thing. You can do a good deal without it, but not everything. Where intuition is combined with exact research it speeds up the progress of research. Exactitude winged by intuition is at times best. But because exact research is exact research, it gets ahead without intuition. It can be logical; it can construct. It can build bridges boldly from one thing to another. It can maintain order in the midst of turmoil.'*

Paul Klee (1928)

*'It has been a source of comfort in troublous times. To gaze up from the ruins of the oppressive present towards the stars is to recognise the indestructible world of laws, to strengthen faith in reason, to realise the harmonia mundi that transfuses all phenomena, and that never has been, nor will be, disturbed.'*

Hermann Weyl (1918)

*'If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties.'*

Francis Bacon – The Advancement of Learning

### Three themes characterise the approach we will adopt

- 1 Generalisation
- 2 Simplification
- 3 Notation and presentation

Through 1&2 you converge on the solution; through 3 you express yourself clearly and concisely. Ideally you develop and generalise the formal techniques you employ while at the same time simplifying the problem in hand in a way that does not jettison its salient features. You should always attempt the simplest problem that you don't understand, rather than trudging through the most complicated problem that you do understand.

So, in the words of a master,

*'The Art of Mathematics is in the identification of the tractable special case that contains within it the germ of generality.'*

D Hilbert



## Numbers

Integers

Addition, subtraction.

Roman and Arabic notation; the latter is one of the greatest steps forward as it

- 1 Systematises that which has gone before
- 2 Makes practical calculation much easier
- 3 Suggests all sorts of generalisation

Multiplication and division. (try doing them in Roman notation)

Subtraction and division, being inverse operations, extend the concept of number from positive integers to negative integers and fractions. (This is all you need to describe anything of importance; quantitative, bottom line stuff)

The irrational number (encouraging the development of Greek geometry; an excellent, physically motivated, theory of the ancient world)

Algebraic notation, representing any number by a symbol; linear, quadratic and other algebraic equations (again an inverse problem)

The concept of an algebraic equation allows us to accommodate irrational numbers but also throws up transcendental (for example  $\pi$  and  $e = 1 + 1/2! + 1/3! + 1/4! + \dots$ ) and imaginary numbers. The extension of the real number system (filling in the real line) underpins differential and integral calculus; we will gloss over this. Complex numbers are widely used the analysis of physical and engineering problems, e.g. the complex exponential representation of harmonic time dependence, ensuring that cause always precedes effect and imposing the condition that we only consider waves radiating away from a source. For the time being we recall that a complex number  $z$  consists of two numbers (its real and imaginary parts)

$$z = x + iy; \quad x \equiv \Re(z), \quad y \equiv \Im(z)$$

which can also be represented in polar form (i.e. amplitude and phase)

$$z = r \exp(i\phi); \quad r = \sqrt{x^2 + y^2}, \quad \tan(\phi) = y/x;$$

$i$  is the 'square root of minus one' whose property  $i^2 = -1$  allows us to extend the usual arithmetic operations from real to complex numbers

$$(a + ib)(c + id) = ac - bd + i(bc + ad)$$

Vectors, tensors etc.

$$\mathbf{y} = \sum_{\alpha} \mathbf{e}_{\alpha} y_{\alpha} \equiv \mathbf{e}_{\alpha} y_{\alpha}$$

$$\mathbf{y} \cdot \mathbf{z} = z_{\alpha} y_{\beta} \delta_{\alpha\beta} = z_{\alpha} y_{\alpha}$$



$$(\mathbf{a} \wedge \mathbf{b})_{\alpha} = \varepsilon_{\varepsilon\beta\gamma} a_{\beta} b_{\gamma}$$

$$\varepsilon_{\alpha\beta\gamma} \varepsilon_{\alpha\mu\nu} = \delta_{\beta\mu} \delta_{\gamma\nu} - \delta_{\beta\nu} \delta_{\gamma\mu}$$

Coordinate systems, Cartesian, spherical polar, cylindrical polar and beyond

So we have got numbers; given one, how do you get to another? (from the question to the answer)

## Functions

Linear

$$y = ax + b$$

Quadratic

$$y = ax^2 + bx + c$$

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Polynomial

$$y = \sum_{r=0}^n a_r x^r$$

Algebraic

$$y = \frac{\sum_{r=0}^n a_r x^r}{\sum_{r=0}^m b_r x^r}$$

Surds and indices

$$y = \sqrt{x}$$

$$y = x^{1/2}$$

$$y = x^{\alpha}$$

Logarithms

$$x = e^{\log(x)}$$

Trigonometry

$$y = \sin(x), \quad \cos(x),$$

$$\sin(x + 2\pi) = \sin(x)$$

$$\cos(x + 2\pi) = \cos(x)$$

$$\cos(0) = 1, \quad \sin(0) = 0$$

De Moivre's theorem

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$\exp(i\pi) + 1 = 0$$

'Hyperbolic' trigonometry

$$e^x = \sinh(x) + \cosh(x)$$



Series expansion

$$y = \sum_{r=0}^{\infty} a_r x^r$$

A few simple examples can get you a long way here; in particular we note the binomial theorem:

$$(1+x)^a = \sum_{n=0}^{\infty} \frac{a(a-1)\cdots(a-n)}{n!} x^n$$

and Taylor's theorem:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0}$$

Attempts to take a theoretical description beyond second order are frequently, but not always, symptoms of your attacking the problem in the wrong way.

## Differential calculus

Time passes and we notice change (or change occurs and time is perceived to pass); rates of change are described by the differential calculus

$$y = y(x) \quad \frac{dy}{dx} = \lim_{\Delta \rightarrow 0} \left( \frac{y(x+\Delta) - y(x)}{\Delta} \right)$$

Simple geometrical ideas about chords and tangents can be made quite rigorous; commonplace observations of the world can be expressed very compactly in terms of derivatives. Thus if  $x(t)$  is the position of an object at time  $t$ , its velocity is given by

$$v(t) = \frac{dx(t)}{dt};$$

its acceleration is given by the second derivative

$$a(t) = \frac{d^2 x(t)}{dt^2}.$$

Physical laws are frequently expressed in terms of the rate of change in a quantity; differential calculus enables us to express the laws as differential equations. Thus, for example, we might have a simple linear 'relaxation' in  $y$  described by

$$\frac{dy}{dt} + \alpha y = 0;$$

the angular displacement of a pendulum obeys

$$\frac{d^2 \theta}{dt^2} + \frac{g}{l} \sin \theta = 0.$$



In Session 2 we will confine our attention to differential calculus with one independent variable (dimension); in Session 3 we will consider generalisations to many dimensions.

## Integral calculus and differential equations

A succession of tiny (infinitesimal) changes result in a finite outcome; how can this be calculated? This bringing together of the many parts to form a whole is the process of integration. The simplest approach to integration is to identify it as the inverse of differentiation; as usual the inverse problem is more difficult than its 'forward' counterpart. A whole set of tricks will have to be learnt, the simplest of which is the use of tables / *Mathematica*. As usual, however, it is amazing how much can be done with just a little background knowledge. Techniques for evaluating integrals that depend on a large parameter, the so-called asymptotic methods, turn out to be very useful. The solution of differential equations such as those describing relaxation and the simple pendulum makes extensive use of methods of the integral calculus. The first has a decaying exponential as a solution. In the small  $\theta$  limit the second differential equation reduces to

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

and can be solved in terms of sines and cosines. (This is an example of the useful trick of linearising an originally non-linear problem; the linear problem is usually something you have seen before.) For larger  $\theta$ , higher order terms become important. Nonetheless it is possible, in this case, to relate  $\theta$  and  $t$  through a single integral; perhaps you would like to have a go at this. It is interesting to note that non-linear equations very like this describe the configuration within a liquid crystal display when it is switched on. In fact this is a nice example of the tractable special case that contains within it the germ of generality.

One dimensional integral calculus will be considered in Session 2; Session 3 will again find us generalising these techniques to higher dimensions

## Matrices, vectors and tensors

When we venture into higher (usually 2 and 3) dimensional spaces we can parameterise these in terms of a variety of co-ordinate systems. Obviously physics is independent of whichever co-ordinate system we adopt to describe it in; we frequently chose a particular co-ordinate system because it simplifies the formal manipulation and/or presents some aspect of the underlying physics in sharp relief. The relationships between different co-ordinate systems will be considered in Session 3. Particular attention will be paid to the description of rotated co-ordinate systems; these allow us to introduce the concepts of vectors, rotation matrices, direction cosines and Cartesian tensors, which in turn streamline much of the vector calculus needed to generalise the results from Session 2 to spaces of higher dimensions. They also enable us to discuss motions of a rigid body (typically a manoeuvring radar platform) clearly and compactly. Finally we consider briefly how the tensor concept can be extended to more general co-ordinate transformations.



## Integral Transforms

Simple functions such as  $\exp(-x)$ ,  $\sin(x)$ ,  $\cos(x)$  and  $x^\alpha$  should be familiar to us all; we may also have a nodding acquaintance with fancier entities such as the Bessel function  $J_\nu(x)$ . To exploit our knowledge of these in a wider context we can represent functions as linear combinations of these more familiar fellows. The Fourier representation of a periodic function as a sum of sines and cosines is a well-known example of this approach. You doubtless also recall that the coefficients in such series are represented as integrals.

Thus we have, for a real, periodic function  $f$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), \quad 0 \leq x < 2\pi; \quad f(x+2\pi) = f(x)$$
$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

In session 4 we will see how these ideas can be extended to aperiodic functions, leading us to the Fourier and Laplace transforms

$$\tilde{f}_F(k) = \int_{-\infty}^{\infty} dx \exp(ikx) f(x)$$
$$\tilde{f}_L(s) = \int_0^{\infty} dt \exp(-st) f(t)$$

we will also consider Hankel and Mellin transforms.

$$\tilde{f}_H(k) = \int_0^{\infty} dx x J_\nu(kx) f(x)$$
$$\tilde{f}_M(k) = \int_0^{\infty} dx x^{k-1} f(x)$$

Perhaps you would like to remind yourselves of Fourier and Laplace transforms of a few simple functions, and their application to the solution of simple differential equations? (This is part of your 'homework'.)

Numerical Fourier transformation plays an important role in much data analysis; we will discuss the relationship between the output of DFT and FFT algorithms and the Fourier transform. We will also give a brief discussion of wavelet and related transforms that are currently finding wider application in data analysis.



## Flash sums demystified (featuring *Mathematica*)

The simple techniques covered in the first few sessions will provide us with much of the toolkit we need when we come to model and analyse radar performance. Nonetheless we will on occasion run into rather more recondite bits and pieces that could, at first sight, put us off. In the fifth session we prepare ourselves for this eventuality in two ways: by flagging up and discussing special topics that crop up in modelling and performance calculations and demonstrating how the *Mathematica* software greatly facilitates the deployment of ‘advanced mathematics’ in the solution of practical problems. In a single session we cannot hope to acquire a detailed background in advanced engineering or physical mathematics, nor yet a reasonable competence in all aspects of what is perhaps the most sophisticated technical software package in the world. Rather more realistically we might be able to dispel some of the mystery that surrounds, for example, gamma, hypergeometric, Legendre, Bessel and other special functions. Untold centuries of mathematician effort have been expended on developing the theory of these special functions; what we hope to do is identify a few useful tools that will enable us to pick our way through the standard texts and tables to the results we want. A great deal of this knowledge is also incorporated into *Mathematica*, and can be accessed quite straightforwardly in graphical and numerical form. *Mathematica* is also able to carry out formal manipulations that would, in bygone days, be prohibitively time consuming for all but the most enthusiastic and unconstrained paper and pencil wielding boffin. If you are not already acquainted with *Mathematica*, you might want to take a quick look at it in your leisure moments. Wolfram’s *Mathematica* book is a bit daunting; Shaw and Tigg’s ‘Applied *Mathematica*: Getting Started, Getting It Done’ provides an interesting, pragmatic and relatively painless introduction.

An example of a special function that crops up in radar modelling is the so-called gamma function

$$\Gamma(z) = \int_0^{\infty} dt \exp(-t) t^{z-1}$$

which allows us to ‘interpolate’ between the integer values of the argument  $z$ , for which it coincides with the factorial function:

$$\Gamma(n+1) = n!.$$

Representing a special function as an integral, as we have done here, allows us to derive many of its properties straightforwardly, as we know those of the simpler functions in the integrand. One of the exercises will lead you through this process for the gamma function. The compound form of the K distribution provides us with a fine example of this approach; you can work out anything you want from

$$P(z) = \frac{b^{\nu}}{\Gamma(\nu)} \int_0^{\infty} dx \frac{\exp(-z/x)}{x} x^{\nu-1} \exp(-bx)$$

without ever having heard of a modified Bessel function of the second kind. Conquering flash sums at the level they crop up at in radar modelling and performance calculations is basically a matter of being foolhardy enough to have a go. As we shall see in the later sessions, a lot of useful results can be conjured forth using the few tricks that we shall discuss in the fifth session; it can also impress the punters something rotten.



## Probability theory, detection, clutter modelling and simulation, performance calculations

The first five sessions of the course should establish the mathematical toolkit required if one is to carry through radar performance and related calculations. The remaining five sessions discuss the principles of radar detection and the modelling of its performance in some depth. The approach adopted is essentially probabilistic; the *a priori* description of the sequence of complex physical events that occur in the radar interrogation of a battlefield or maritime scenario is both too difficult and time-consuming and too burdened with superfluous detail to be of any practical use. If, however, we can characterise the returns from a target and its environment in terms of probability density functions we can make use of statistical decision theory to ascertain the likely origin of a given return. Operational useful figures of merit such as probabilities of detection and false alarm can also be calculated on the basis of such models. To provide a background for this approach we will review the elements of probability theory and stochastic processes in the sixth session. Once again we find ourselves reviewing a potentially vast subject area; we will attempt to focus our attention on those aspects that are directly relevant to radar algorithm design and performance modelling. As well as single point statistics we will consider the correlation properties of returned signals, including their descriptions in terms of power spectra and stochastic differential equations.

Once we have introduced these elements of the probabilistic description of the signals we will, in the seventh session, discuss how they form the basis of likelihood ratio based detection procedures that are, in some sense, optimal. These provide a starting point for the construction of practical detection algorithms; this can involve a trade off between their practical implementability and the extent to which they approach the theoretical ideal. The problems of detection are closely related to those of estimating parameters of a statistical model of a given set of data. When the temporal evolution of a data set is modelled by a stochastic differential equation, this problem of estimation is essentially that of tracking. The Kalman filter will be discussed in this context; more attention will be paid to its principles than to the details of practical implementation.

In some circumstances we are faced with the problem of detecting a small target in a significant clutter background. Under these circumstances a realistic and tractable clutter model is required if we are to make progress in the development of effective detection procedures. In the early days of radar, clutter was invariably modelled as a Gaussian process. This model had the advantages of analytical tractability and, as a consequence of the central limit theorem, wide-spread applicability to low resolution radar systems. With the development of high resolution radar systems, prompted by the desire to raise the signal to clutter ratio and so enhance detectability, it was found that this simple model was no longer valid. The central limit theorem had broken down; the clutter was out of spec. The modelling of non-Gaussian clutter presented very significant problems. Until 1980 *ad hoc* empirical models, such as the log-normal and Weibull statistics, were used; these however had little if any physical foundation and so were of little assistance in the development of detection strategies. The introduction of the K distribution model changed all this; at once it incorporated a reasonable physical picture of the processes generating the spiky non-Gaussian clutter and at the same time highlighted ways in which detection procedures could be improved. After twenty years or so, some deficiencies in the K model are becoming more apparent. Nonetheless its underlying phenomenological basis continues to provide a framework in which these can be remedied. The development of the K distribution and related clutter models will be described in the eighth session.

In this course considerable emphasis is laid on direct derivation and calculation. Nonetheless there are many practically important problems that might not submit to such frontal analytic attack and are better investigated by simulation. Realistic models for clutter returns must play



a central role in such studies. Given the success of the K distribution in modelling non-Gaussian clutter, one would expect a numerically generated K process with controlled spectral and correlation properties to provide such a class of models. In the ninth session we discuss the problems presented by the simulation of correlated random processes. When the process is Gaussian, reasonable control over the correlation properties can be established without too much difficulty. Significantly greater difficulties are encountered in the non-Gaussian case, and it is only recently that some of the problems encountered in the simulation of correlated K distributed and other non-Gaussian processes have been overcome. We will discuss these and related developments in some detail.

## Textbooks

You must search for the books that you, personally, find most useful. Those I learnt from as a boy are very old now; nonetheless I use some of them to this day. I would suggest that you ensure that you have access to Abramowitz and Stegun or Gradshteyn and Ryzhik; alternatively you can experiment using *Mathematica*. These all provide access to useful properties of elementary and special functions, along (with the exception of G&R) with their numerical values. When you use a result from one of these sources you should convince yourself that it is right (or find that it is wrong). Beyond this it is up to you. I would not, however, suggest that you dive into one of the seriously 'hard core' tomes like Whittaker and Watson's *Modern Analysis*. Engineer's introductions are much more useful for a first stab at a subject. If you can't understand the presentation of a topic in a given textbook the fault is as likely to be its author's as it is to be your own. So, look around. Find out what your colleagues learnt from. For what it's worth the following have provided me with the basis of my 'mathematical' self-education up to this point

R.I. Porter, '*Further elementary analysis*' and '*Further Mathematics*', G. Bell and Sons, London, 1962

H.S. Hall and S.R Knight, '*Higher Algebra*', MacMillan, 1887

W.D. Day, '*Introduction to vector analysis for radio and electronic engineers*', Iliffe Books, 1966

C.A. Coulson, '*Valence*', Clarendon Press, Oxford, 1961

T.M. MacRobert, '*Functions of a complex variable*', MacMillan, London, 1962

G. Stephenson, '*Mathematical methods for science students*', Longmans,

L. Fox and D.F. Mayers, '*Computing methods for scientists and engineers*', Clarendon Press, Oxford, 1968

P.M. Morse and H. Feshbach, '*Methods of theoretical physics; Vols. 1&2*', McGraw Hill. 1953

H. and B.S. Jeffreys, '*Methods of mathematical physics*', Cambridge University Press, 1966

M. Abramowitz and I.A. Stegun, '*Handbook of mathematical functions*', Dover, New York, 1965

I.S. Gradshteyn and I.M. Ryzhik, '*Table of integrals, series and products*', Academic Press, New York, 1965

B. Spain, '*Tensor Calculus*', Oliver and Boyd, 1960



- I.N. Sneddon, '*Special functions of mathematical physics and chemistry*', Oliver and Boyd, 1961
- M.R. Spiegel, Schaum Outline Series,
- E.T. Copson, '*Theory of functions of a complex variable*', Clarendon Press, Oxford, 1935
- E.T. Whittaker and G.N. Watson, '*A course of modern analysis*', Cambridge University Press, 1927
- E.L. Ince, '*Ordinary differential equations*', Longmans, 1926
- J.A. Stratton, '*Electromagnetic theory*', McGraw Hill, New York, 1941
- M Hamermesh, '*Group theory and its application to physical problems*', Addison Wesley, New York, 1964
- D.M. Brink and G.R. Satchler, '*Angular momentum*', Clarendon Press, Oxford, 1968
- J.D. Tolman, '*Special functions: a group theoretic approach*', Benjamin, New York, 1968
- F.W.J. Olver, '*Asymptotics and special functions*', Academic Press, New York, 1974
- N. Wax (Ed.), '*Selected papers on noise and stochastic processes*', Dover, New York, 1954
- D. Middleton, '*An introduction to statistical communication theory*', McGraw Hill, New York, 1960
- S. Wolfram, '*Mathematica: a system for doing mathematics by computer*', (2nd Edn.), Addison Wesley, Redwood City, 1991
- W.T. Shaw and J. Tigg, '*Applied Mathematica: getting started, getting it done*', Addison Wesley, 1994
- If, at the end of these sessions, you can make up a list that you feel as comfortable with then significant progress will have been made.



## Exercises

Have a go at some of these if you like. Remember that this is not a competition, and expect the level of difficulty to fluctuate wildly, even within a single exercise.

1. Recall the following formula for the sum of a geometric series

$$\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}; |x| < 1$$

How is this modified when you have a finite number of terms? Hence, or otherwise show that

$$\sum_{r=0}^{2^n-1} x^r = \prod_{k=0}^{n-1} (1+x^{2^k}) \quad \clubsuit$$

What is the relevance of this result to the binary representation of integers and to your Granny's kitchen scales?

When Feynman was a (very little) boy he discovered that  $\cos(20^\circ)\cos(40^\circ)\cos(80^\circ)$  has a surprisingly simple value and was delighted that he could explain why. Can you reproduce and generalise this early manifestation of genius? (Hint: find out what this simple value is by direct numerical calculation (the easy part), express the cosines in terms of exponentials using De Moivre's theorem, and use the result  $\clubsuit$ )

2. A pendulum undergoes simple harmonic motion which, in the limit of a small angular displacement, is described by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

The period of this motion can be written down by inspection; what is it? Sadly we can't do the same thing for the finite amplitude oscillation satisfying the non-linear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

Consider an alternative analysis of the simple harmonic motion case; this is deliberately brief so that you can fill in the details:

$$\begin{aligned} \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta &= 0 \\ \frac{d\theta}{dt} \left( \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta \right) &= 0 \\ \frac{d}{dt} \left( \left( \frac{d\theta}{dt} \right)^2 + \frac{g}{l}\theta^2 \right) &= 0 \\ \frac{d\theta}{dt} &= \sqrt{\frac{g}{l}(\theta_m^2 - \theta^2)} \\ T &= 4 \sqrt{\frac{l}{g}} \int_0^{\theta_m} \frac{d\theta}{\sqrt{\theta_m^2 - \theta^2}} = 2\pi \sqrt{\frac{l}{g}} \end{aligned}$$



Once you are happy with this approach adapt it to the finite amplitude pendulum, taking it as far as you can.

- 3 What are the Laplace transforms of the first and second derivatives of the function  $y(t)$ ? Use this result to obtain the Laplace transform of  $y(t)$  if it satisfies the differential equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = \exp(-\gamma t) \quad t \geq 0$$

Note how use of the Laplace transform 'builds in' the initial conditions that define the required solution. Invert the Laplace transform to get  $y(t)$ . Discuss the answers you get, and their possible physical interpretation. Under what circumstances would the Fourier transform be a better tool with which to tackle 'driven' differential equations of this type? Give an example if you feel like it.

- 4 A few properties of the gamma function. Starting from the definition

$$\Gamma(z) = \int_0^{\infty} dt \exp(-t) t^{z-1}$$

show that

$$z\Gamma(z) = \Gamma(z+1)$$

$$\Gamma(n+1) = n! \text{ integer } n$$

Relate  $\Gamma(1/2)$  to the 'well known' Gaussian integral  $\int_0^{\infty} dx \exp(-x^2)$ ; using this result evaluate  $\Gamma(-3/2)$  and  $\Gamma(5/2)$ . Prove that

$$(2n)! = 2^{2n} n! \Gamma(n+1/2) / \sqrt{\pi}$$

The pdf of a K distributed intensity  $z$  has the form

$$P(z) = \frac{b^\nu}{\Gamma(\nu)} \int_0^{\infty} dx \frac{\exp(-z/x)}{x} x^{\nu-1} \exp(-bx)$$

Evaluate the  $n$ th moment of this i.e.

$$\langle z^n \rangle = \int_0^{\infty} dz z^n P(z).$$

Can you make any progress with the evaluation of the 'characteristic function'

$$C(s) = \int_0^{\infty} dz \exp(-sz) P(z) ?$$



5 *Mathematica* version 2.1 gave the following answer:

$$\int_0^1 x^{\alpha+1} \frac{(1+x-2x^\alpha)}{(1-x^2)} dx = 0$$

for general  $\alpha$ ; this is obviously wrong (Why?). Strangely enough, if we put in specific integer or rational values of  $\alpha$ , and waited long enough, *Mathematica* 2.1 gave the right answers in these special cases, e.g.

$$\int_0^1 x^3 \frac{(1+x-2x^2)}{1-x^2} dx = \log 2 - \frac{1}{3}; \quad \int_0^1 x^4 \frac{(1+x-2x^3)}{1-x^2} dx = \log 2 - \frac{1}{4} \quad \int_0^1 x^8 \frac{(1+x-2x^7)}{1-x^2} dx = \log 2 - \frac{1}{8}$$

Use these to guess the general result, then prove it. This can be done in about three lines, using no more than rusty A level and low cunning. It does help to know the answer though. (Later versions of *Mathematica* get it right anyway; you still have to prove it though) Should anyone have a working knowledge of the properties of polygamma functions (Abramowitz and Stegun, Section 6.3 ff) they might well be able to chart a much less scenic route to the hoped for result.